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Poisson Structures due to Lie Algebra Representations

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Using a unitary solution of the classical Yang–Baxter equation on a Lie algebra $\mathcal{G}$ we describe a particular way of constructing homogeneous quadratic Poisson structures on the dual of a $\mathcal{G}$-module V and study some local features of the symplectic foliation due to the involutive distribution of the Hamiltonian vector fields. We also give some examples where the symplectic leaves are explicitly calculated.

1. INTRODUCTION

Poisson brackets on phase spaces play a crucial role in Hamiltonian mechanics and quantization. This led mathematicians to the study of Poisson manifolds. A Poisson manifold M is a smooth manifold endowed with an alternating 2-derivation $\{\cdot,\cdot\}$ on the algebra of smooth functions on it such that $\{\cdot,\cdot\}$ is a Lie bracket. The bivector field P defined by $\{\cdot,\cdot\}$ on M is called a Poisson structure on M. Hence, for any smooth function f on M we can associate a vector field X_f on M defined by $X_f(g) = -\{f, g\}$ for any smooth function g on M. These vector fields form an involutive distribution and give a foliation of M in which each leaf is symplectic. These are some of the important features of Poisson manifolds.

Recall. 1). If $X_1, X_2, \ldots, X_n$ is a basis of a Lie algebra $\mathcal{G}$, then $R = \sum_{i,j=1}^{n} r_{ij}X_i \otimes X_j$ is called a unitary solution of the classical Yang–Baxter equation (CYBE) on $\mathcal{G}$ (Drinfeld, 1983) if R is skew-symmetric and

$$\sum_{i,j,k,l} r_{ij}r_{kl}([X_i, X_k] \otimes X_j \otimes X_l + X_i \otimes [X_j, X_k] \otimes X_l + X_i \otimes X_k \otimes [X_j, X_l]) = 0$$

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2. If T: $V \to V$ is an endomorphism, then one can define a vector field $\tilde{T}$ on V^* by $\tilde{T}(f)(\alpha) = \langle \alpha, T(df(\alpha)) \rangle$, where $f \in C^\infty(V^*)$ and $\alpha \in V^*$ (here we identify V with V^{**}) (Bhaskara and Viswanath, 1988).

Let $v_1, v_2, \ldots, v_n$ be a basis of a vector space V and let $x_1, x_2, \ldots, x_n$ be the corresponding coordinate system on V^*. The endomorphisms T_{ij}: $V \to V$ defined by $T_{ij}(v_k) = \delta_{ik} v_j$ form a basis for End(V) and define vector fields $\tilde{T}_{ij}$ on V^*, where $\tilde{T}_{ij} = x_j\, \partial/\partial x_i$.

It is proved in Bhaskara and Rama (1991) that if

$$R = \sum_{i,j,k,l=1}^{n} r_{ij}^{kl} T_{ij} \otimes T_{kl}$$

is a unitary solution of the CYBE on End(V), then

$$\tilde{R} = \sum_{i,j,k,l=1}^{n} r_{ij}^{kl} \tilde{T}_{ij} \otimes \tilde{T}_{kl}$$

is a Poisson structure on V^*.

A simple calculation gives that

$$\tilde{R} = \sum r_{ij}^{kl} x_j x_l \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_k}$$

and hence the Poisson structure is a quadratic homogeneous Poisson structure (HQPS).

In this paper we use a unitary solution of the CYBE on $\mathcal{G}$ to describe a particular way of constructing HQPS on the dual of a $\mathcal{G}$-module V and study some local features of the symplectic foliation due to the involutive distribution of the Hamiltonian vector fields. We also give some examples where the symplectic leaves are explicitly calculated as the level sets of Casimir functions.

2. POISSON STRUCTURES DUE TO REPRESENTATIONS OF $\mathcal{G}$

Let $\mathcal{G}$ be a real, finite-dimensional Lie algebra and let ϕ be a finite-dimensional representation of $\mathcal{G}$ in V. Consider a unitary solution R of the CYBE on $\mathcal{G}$. Then $\phi(R)$ is a unitary solution of the CYBE on End(V) and $\widetilde{\phi(R)}$ is an HQPS on V^*. Let us call it the Poisson structure due to ϕ on V^* (of course, it depends on R). Once R is fixed, each representation leads to an HQPS. The Poisson bracket of this structure is written as $\{\cdot\,, \cdot\}_R$, showing its dependence on R. In what follows, we fix a unitary solution R of the CYBE on $\mathcal{G}$ and all the representations considered are finite dimensional.

Theorem 1. If ϕ and ψ are two equivalent representations of $\mathcal{G}$ in V and W, respectively, then their corresponding Poisson structures are isomorphic.

Proof. Since ϕ and ψ are equivalent representations, there exists an isomorphism λ: $V \leftrightarrow W$ such that $\mathrm{End}\lambda \circ \psi = \phi \circ id$, where $\mathrm{End}\lambda$: $\mathrm{End}(W) \to \mathrm{End}(V)$ is defined by

$$\mathrm{End}\lambda(T_W)(v) = \lambda^{-1}(T_W(\lambda(v)))$$

for all $T_W \in \mathrm{End}(W)$ and $v \in V$. Denote by μ the dual map λ^* of λ. We can define μ_*: $\chi(W^*) \to \chi(V^*)$ by $\mu_*(X_W)(f) = X_W(f \circ \mu) \circ \mu^{-1}$.

Now,

$$\begin{aligned}\mu_*(X_W)(f)(\alpha) &= \widetilde{T_W}(f \circ \mu) \circ \mu^{-1}(\alpha)\\ &= \langle \mu^{-1}(\alpha), T_W(df \circ d\mu)(\mu^{-1}(\alpha))\rangle\\ &= \langle(\alpha), \lambda^{-1}T_W(\lambda(df(\alpha)))\rangle\\ &= \widetilde{\mathrm{End}\lambda(T_W)}(f)(\alpha)\end{aligned}$$

and hence we can prove that

$$\Lambda^2\mu(\widetilde{\psi(R)}) = \widetilde{\phi(R)}$$

This shows that the Poisson structures are isomorphic.

Theorem 2. Let ϕ be a representation of $\mathcal{G}$ in V such that $\phi = \Sigma_i\phi_i$, where each ϕ_i is an irreducible representation on $\mathcal{G}$ in V_i and $V = \oplus_{i=1}^k V_i$, where $k \leq \dim V$. Then the Poisson structure due to ϕ is exactly the sum of the Poisson structures due to the ϕ_i.

Proof. Let B_i be a basis of $\mathrm{End}(V_i)$ for $i = 1, \ldots, k$. Let us extend the elements of B_i to $\mathrm{End}(V)$ by declaring them to be zero on V_j, where $j \neq i$. Call this set of extended elements $\overline{B_i}$. Now $B = \{\overline{B_i}\}_{i=1,\ldots,k}$ is part of a basis of $\mathrm{End}(V)$. However, B has the following property:

(*) If $T \in \overline{B_i}$ and $S \in \overline{B_j}$, then $[T, S] = 0$ for all $i \neq j$.

Since $\phi = \Sigma_{i=1}^k \phi_i$, each $\phi(X)$ for $X \in \mathcal{G}$ is an endomorphism of V which can be expressed as a linear span of elements of B only. Obviously each ϕ_i, treated as an endomorphism of V, is a linear combination of elements of $\overline{B_i}$.

Denote $\widetilde{\phi_i(R)}$ and $\widetilde{\phi(R)}$ by P_i and P, respectively. P_i and P are HQPSs on V_i^* and V^*, respectively, and each P_i can also be treated as a Poisson structure on V^*. By property (*) one can prove that $[P_i, P_j] = 0$ (here $[\cdot\,, \cdot]$ is the Schouten bracket (Bhaskara and Viswanath, 1988; Lichnerowicz, 1979) for all $i \neq j$ and hence $\Sigma\, P_i$ is an HQPS on V^*. Since $\phi(R) = \Sigma_{i=1}^k\, \phi_i(R)$, we get the required result.

By the above two theorems we can conclude the following: (a) It is enough to study the Poisson structures due to any one representative of an equivalence class of representations, (b) if a representation is completely reducible, then it is enough to study the Poisson structures due to its irreducible components and (c) the Poisson structures due to equivalent representations give rise to isomorphic symplectic foliations.

Representations of the semisimple Lie algebras supply a large class of examples, as they are completely reducible.

Example 1. Suppose $\mathcal{G} = sl(2, \Re)$ with its standard basis X, H, Y with the Lie bracket relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H$$

We know the description of all irreducible representations. Let V be an $(r + 1)$-dimensional vector space with a basis $v_0, \ldots, v_r$ and let T_{ij} be the basis of End(V) defined by the above basis of V. If ϕ is an irreducible representation of $sl(2, \Re)$ in V, then $\phi(H)$, $\phi(X)$ are represented by the diagonal matrix $\{r, r - 2, \ldots, -r\}$ and the lower off-diagonal matrix $\{r, r - 1, \ldots, -r\}$, respectively, with respect to the basis of V. Hence

$$\phi(H) = \sum_{i=0}^{r} (r - 2i)T_{ii}, \qquad \phi(X) = \sum_{j=0}^{r} (r - j + 1)T_{jj-1}$$

where for $j = 0$ we understand that $T_{jj-1} = 0$. Now consider $R = H \wedge X$, a unitary solution of the CYBE on $\mathcal{G}$. Then

$$\widetilde{\phi(R)} = \widetilde{\phi(H)} \wedge \widetilde{\phi(X)}$$
$$= \sum_{i,j=0,\ldots,r} (r - 2i)(r - j + 1)x_i x_{j-1} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

Therefore the Poisson brackets are given by

$$\{x_k, x_l\}_R = (r - 2k)(r - l + 1)x_k x_{l-1} - (r - 2l)(r - k + 1)x_l x_{k-1}$$

for $k, l = 0, \ldots, r$, where we define $x_{-1} = 0$.

3. CASIMIR FUNCTIONS AND THE SYMPLECTIC FOLIATION

Let $V = \oplus_{i=1}^{k} V_i$ and let $P = \Sigma_i P_i$ be a Poisson structure on V where each P_i is a Poisson structure on V_i for each i. Choose a coordinate system on each V_i. Then all these coordinate systems together form a coordinate system on V. It is easy to see that rank $P(x) = \Sigma_i$ rank $P_i(x_i)$ where $x = \Sigma_i x_i$.

The set $D = \{x \in V$: $P(x)$ has the maximal rank$\}$ is open and dense in V. We can choose a neighborhood U of $x = \Sigma_i x_i$ in V such that $U = \Omega_{i=1}^{k} U_i$ where each U_i is open in V_i.

Let $\{f^i_{\kappa_i}\}_{\kappa_i}$ be a maximal independent set of Casimir functions on the Poisson manifold (U_i, P_i). Then $\{\{f^i_{\kappa_i}\}_{\kappa_i}\}_i$ form a maximal independent set of Casimirs on (U, P).

Since locally the symplectic leaves are given by the common level sets of Casimir functions, around a point x in D we can describe the symplectic leaves of (U, P) if we know the symplectic leaves of each (U_i, P_i). This is not the case when $x \notin D$.

Let R be a unitary solution of CYBE on $\mathcal{G}$. We denote by $\{\cdot\,,\cdot\}_L$ the linear (or Lie–) Poisson structure on $\mathcal{G}^*$ and by $\{\cdot\,,\cdot\}_R$ the HQPS on $\mathcal{G}^*$ due to the *ad*-representation. Then any Casimir with respect to $\{\cdot\,,\cdot\}_L$ is a Casimir with respect to $\{\cdot\,,\cdot\}_R$. Indeed, choose a basis $\{X_i\}_i$ of $\mathcal{G}$ and the coordinate system $\{x_i\}_i$ defined by it on $\mathcal{G}^*$. If $R = \Sigma_{i,j} r_{ij} X_i \otimes X_j$ with $r_{ij} = -r_{ji}$, then for any $f, g \in C^\infty(\mathcal{G}^*)$ we have

$$\{f, g\}_R = \sum_{i,j} r_{ij} \{f, x_i\}_L \{x_j, g\}_L$$

This proves the claim.

An interesting consequence of the above claim occurs when the dimension of the Lie algebra is three. For any R the HQPS due to the *ad*-representation and the Lie–Poisson structure have the same symplectic leaves around any point at which both the structures have the maximal rank. This is true because the dimension of any symplectic leaf with respect to any structure must be two. Hence by the above discussion we can conclude that there exists a Casimir which describes the symplectic leaves of both structures around the point. In the following examples one can observe this fact.

All the results and the discussions in this article conclude that it is enough to know the description of the symplectic leaves (at least locally) of the HQPSs due to irreducible representations on the dual of a Lie algebra module. This will enable us to describe the symplectic foliation locally around any point of an open dense subset of the direct sum of the duals of the irreducible modules. In what follows we discuss the Casimirs of some HQPSs due to the adjoint representation.

Example 2. 1. Let $\mathcal{G} = \mathcal{G}l(2, \mathfrak{R})$ with the standard basis $\{X_1, X_2, X_3, X_4\}$, where each is a 2×2 matrix with all the entries equal to zero except that the (1×1) entry in X_1, the (1×2) entry in X_2, the (2×1) entry in x_3, and the (2×2) entry in X_4 are ones. It is easy to check that $R = X_1 \wedge X_2$ is a unitary solution of the CYBE on $\mathcal{G}$. Let $\{x_1, x_2, x_3, x_4\}$ be the coordinate system on $\mathcal{G}^*$ defined by $\{X_1, X_2, X_3, X_4\}$. Then the HQPS on $\mathcal{G}^*$ due to the *ad*-representation is given by

$$\{x_1, x_2\}_R = x_2^2, \qquad \{x_1, x_3\}_R = -x_2x_3$$
$$\{x_1, x_4\}_R = 0, \qquad \{x_2, x_3\}_R = x_1x_2 - x_2x_4$$
$$\{x_2, x_4\}_R = x_2^2, \qquad \{x_3, x_4\}_R = -x_2x_3$$

The plane $\{x_2 = 0\}$ is the set of singularities of the above Poisson structure. Hence each point of this plane is a symplectic leaf of dimension zero. The two Casimir functions are $x_2x_3 - x_1x_4$ and $x_1x_2x_3 + x_2x_3x_4 - x_1x_4^2 - x_1^2x_4$.

2. Let $\mathcal{G} = so(3, \Re)$ with the basis $\{X_1, X_2, X_3\}$ with which the Lie bracket relations are as follows:

$$[X_1, X_2] = X_3, \qquad [X_2, X_3] = X_1, \qquad [X_3, X_1] = X_2$$

It is easy to check that $R = X_1 \wedge X_2$ is a unitary solution of the CYBE on $\mathcal{G}$. Let $\{x_1, x_2, x_3\}$ be the coordinate system on $\mathcal{G}^*$ defined by $\{X_1, X_2, X_3\}$. Then the HQPS on $\mathcal{G}^*$ due to the *ad*-representation is given by

$$\{x_1, x_2\}_R = x_3^2, \qquad \{x_2, x_3\}_R = x_1x_3, \qquad \{x_3, x_1\}_R = x_2x_3$$

The plane $\{x_3 = 0\}$ is the set of singularities of the Poisson structure. Hence each point of this plane is a symplectic leaf of dimension zero. $x_1^2 + x_2^2 + x_3^2$ is the Casimir.

3. Let $\mathcal{G} = sl(2, \Re)$ with the basis $\{X, H, Y\}$ with which the Lie bracket relations are as follows:

$$[H, X] = 2X, \qquad [X, Y] = H, \qquad [H, Y] = -2Y$$

It is easy to check that $R = H \wedge X$ is a unitary solution of the CYBE on $\mathcal{G}$. Let $\{x, h, y\}$ be the coordinate system on $\mathcal{G}^*$ defined by $\{X, H, Y\}$. Then the HQPS on $\mathcal{G}^*$ due to the *ad*-representation is given by

$$\{h, x\}_R = 4x^2, \qquad \{x, y\}_R = 2hx, \qquad \{h, y\}_R = -4xy$$

The plane $\{x = 0\}$ is the set of singularities of the Poisson structure. Hence each point of this plane is a symplectic leaf of dimension zero. $xy + h^2$ is the Casimir.

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